Dynamical theory of quantum chaos or a hidden random matrix ensemble?

In a recent Letter [1] Altland and Zirnbauer claim that they rigorously proved the *complete* analogy between a (classically chaotic) dynamical system and disordered (random) solids. The main purpose of this Comment is to show that,in fact, the theory [1] does not represent at least some characteristic dynamical features of the quantum kicked rotator (QKR), an example chosen in [1] for comparison with the theory.

The phenomenon of quantum suppression of classically chaotic diffusion was pointed out long ago [2]. Later on, a large number of papers appeared in which this phenomenon was numerically, analytically and even experimentally [3] confirmed. In particular, an important step was the correspondence found between the QKR and the 1D disordered model [4, 5]. However, in spite of this and in spite of all mathematical efforts , so far, a rigorous proof of quantum localization in the QKR is still lacking. The main mathematical difficulty lies in the fact that, for a rigorous proof of localization, the theoretic number properties of the period τ are crucial. As a matter of fact, rigorous statements can be made only in the opposite direction. Indeed, it was analytically shown [6] that for rational values of the parameter $T = \tau/4\pi$ no localization takes place (the so-called "quantum resonance"). Moreover, it was rigorously proven [7] that localization is absent also for a non-empty set of irrational values of T. It is not known how 'large' is this set (even if we may guess that its Lebegue measure is zero) and if and how it depends on the perturbation strength k. Even for a typical (with probability 1) irrational T, when the whole everywhere dense set of quantum resonances is still insufficient to destroy localization, the eigenfunctions may be, nevertheless, essentially modified if the detuning from resonance is small enough. The measure of such T is typically small but finite [6]. As far as we understand, the paper [1] does not take into account this problem which constitutes an important feature and difficulty of the QKR.

In our opinion, the important dynamical features mentioned above were lost in the

approximate calculations following the exact functional (6) in ref. [1]. The following simple argument can demonstrate our main point. By direct averaging over the parameter ω_0 (we use the same notations as in [1]) one easily finds

$$Q_{1,2;2,1}(\omega) \equiv \langle\, \langle l_1|G^+(\omega_+)|l_2\rangle\, \langle l_2|G^-(\omega_-)|l_1\rangle\, \rangle_{\omega_0} = \sum_{n=0}^{\infty} \exp[in(\omega\tau + i0)]\, |\langle l_1|U^n|l_2\rangle|^2\ . \tag{1}$$

This <u>exact</u> expression contains matrix elements of the dynamical Floquet operator U so that all dynamical peculiarities of QKR survive the ω_0-averaging. We reckon that eq. (6) in ref. [1] gives an integral representation of the quantity (1) and therefore carries the same full dynamical information. Let us now consider, for example, the simplest quantum resonance with $\tau = 4\pi$. In this case $\langle l_1|U^n|l_2\rangle = \langle l_1|\exp[-i(nk\, cos\theta)]|l_2\rangle$ and (1) reduces to

$$Q_{1,2;2,1}(\omega) = \sum_{n=0}^{\infty} \exp[in(\omega\tau + i0)]\ J^2_{|l_1-l_2|}(nk) \tag{2}$$

where J_l stands for the Bessel function. Using cosine asymptotics of the Bessel function one readily estimates the singular part of (2) at the infrared limit $\omega \to 0$ as

$$Q_{1,2;2,1}(\omega) = \frac{2}{\pi k}\ \ln \frac{i}{\omega\tau + i0} \tag{3}$$

Contrary to the localization regime, the r.h.s in (3) does not depend on the distance $|l_1 - l_2|$. Such resonance regimes of motion (as well as the nearly resonant values of T) are not reproduced by the calculation reported in [1]. They "mysteriously" disappear from the theory between eqs. (6) and (8). This leads us to suspect that some hidden additional statistical assumption has been made in [1] which remains uncleared. Most likely, the very structures of the principal configurations in (6) must be quite different depending on how close to rationals is parameter T. This delicate problem is completely ignored in [1].

An additional remark concerns the dependence on the "symmetry breaking" parameter a. It is trivially seen that the introduction of this parameter results only in the phase transformation

$$\langle l_1|U^n|l_2\rangle \to \exp[i(l_1 - l_2)\, a]\ \langle l_1|U^n|l_2\rangle \tag{4}$$

of the matrix elements of an arbitrary power n of the Floquet operator. Therefore, this parameter completely disappears from eq. (1) and by no means can influence the localization length. The presence of this parameter in the final expression (9) in Ref. [1] is fully due to the angular discretization used. An analysis of the dependence of the localization length on symmetry breaking parameters in the generalized QKR is given in [8].

Finally, it is probably worth just mentioning the kicked Harper model which exhibits the same classical diffusion as the kicked rotator but which has an extremely rich quantum behaviour; in particular, delocalization can take place for any value, rational or irrational, of the kick period T. As discussed in [9] this model clearly shows how subtle is the problem to understand the features of classical chaotic dynamics which lead to quantum diffusion or localization. This problem remains open.

In conclusion, the theory [1], in its present form may be expected to be valid when standard band random matrix approach can be ad hoc used. However, it fails to take into account dynamical features which go beyond the random matrix theory description.

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Giulio Casati

International Center for the Study of Dynamical Systems
University of Milano at Como
Via Castelnuovo, 7 - 22100 COMO Italy

F.M.Izrailev and V.V.Sokolov

Budker Institute of Nuclear Physics
630090 Novosibirsk, Russia